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Critical Properties of Spin-Glasses

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Abstract
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Critical Properties of Spin-Glasses*

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The critical properties of the model of a spin-glass proposed by Edwards and Anderson are studied using the renormalization group. The critical exponents are calculated in $6-\epsilon$ spatial dimensions. It is argued that a tricritical point can exist where the nonordering field is the skewness of the distribution of J.

Although spin-glasses, such as dilute solutions of Mn in Cu,[1,2] have been studied experimentally for many years, only recently have formulations been given in terms of a microscopic Hamiltonian.[3-6] Even so, the spin-glass transition has not been successfully related to the usual picture of phase transitions as we shall do here. As in Refs. 3–6 we consider the spin Hamiltonian, $\mathcal{H}$, given by

$$\mathcal{H}/kT = -\sum_{r,r'} K(\vec{r},\vec{r}\,')\vec{S}(\vec{r})\cdot\vec{S}(\vec{r}\,'), \qquad (1)$$

where $\vec{S}(\vec{r}) = S_1(\vec{r}), S_2(\vec{r}), \ldots, S_m(\vec{r})$ is a classical m-component spin of unit magnitude at the lattice point $\vec{r}$, and $K(\vec{r},\vec{r}\,') = J(\vec{r},\vec{r}\,')/kT$, where $J(\vec{r},\vec{r}\,')$ is a random variable with a probability distribution $P(\vec{r},\vec{r}\,';J)$, and $J(\vec{r},\vec{r}\,')$ is assumed to be a finite-ranged interaction. We treat a quenched random system where the average free energy is calculated as the average, denoted $[\,]_{\rm av}$, over all configurations of $J(\vec{r},\vec{r}\,')$:

$$F = [F(\{J\})]_{\rm av}. \qquad (2)$$

According to mean-field theory one expects a ferromagnetic or antiferromagnetic state if $[J(\vec{r},\vec{r}\,')]_{\rm av}$ is sufficiently large in magnitude. If $[J(\vec{r},\vec{r}\,')]_{\rm av}$ is zero, Edwards and Anderson (EA)[3] argue that there will still be a transition at a freezing temperature T_f to an ordered state characterized by a new order parameter,

$$q(\vec{r}) = [\langle\vec{S}(\vec{r})\rangle_{\{J\}}\cdot\langle\vec{S}(\vec{r})\rangle_{\{J\}}]_{\rm av}, \qquad (3)$$

where $\langle\vec{S}(\vec{r})\rangle_{\{J\}}$ is the thermal average of $\vec{S}(\vec{r})$ for a given configuration $\{J\}$. Note that q is by definition a positive quantity. This will be important in what follows. EA calculate the properties of this spin-glass phase transition using mean-field theory and a Gaussian random distribution of J's centered about $[J(\vec{r},\vec{r}\,')]_{\rm av} = 0$. They find a continuous transition with an order-parameter exponent of $\beta = 1$ and a finite discontinuity in the slope of the specific heat, $dC(T)/dT$, at $T = T_f$, so that $\alpha = -1$. Similar results were found by other more detailed calculations.[4-6] A straightforward generalization of the EA treatment to include an external field conjugate to the order parameter yields susceptibility and correlation-length exponents $\gamma=1$ and $\nu=\frac{1}{2}$. These exponents should be valid for spatial dimensionality, d, greater than a critical value, d_c. The value of d_c may be determined as the value of d for which the scaling relation $2\beta+\gamma=d_c\nu$ is satisfied by the mean-field values of the exponents given above. Thus $d_c=6$ for the spin-glass problem. The same argument was used by Toulouse[7] to correctly predict $d_c=6$ for the percolation problem[8] and by other authors in connection with similar random systems.[9,10] Deviations from mean-field theory occur for $d<d_c$ and are of order $d_c-d=\epsilon$ for $d\to d_c^-$.

In this paper we will use the renormalization group[11,12] (RG) to analyze the spin-glass transition. As predicted above, this analysis will lead to an ϵ expansion in $6-\epsilon$ dimensions. We begin with a Hamiltonian of Eq. (1) with only nearest-neighbor interactions $K(\vec{r},\vec{r}+\vec{\delta})$. As is convenient in treating random systems,[3] we evaluate the partition function of the system replicated n times:

$$Z^{(n)}=\int d\{K\}P(\{K\})\,\mathrm{Tr}_s\exp(-\mathcal{H}^{(n)}/kT), \qquad (4)$$

where

$$\frac{\mathcal{H}^{(n)}}{kT}=-\sum_{\alpha=1}^{n}\sum_{\vec{r},\vec{\delta}}K(\vec{r},\vec{r}+\vec{\delta})\vec{S}^\alpha(\vec{r})\cdot\vec{S}^\alpha(\vec{r}+\vec{\delta}) \qquad (5)$$

and where Tr_s is a trace over all spin variables. The free energy of Eq. (2) is then given by

$$F=\lim_{n\to 0}(-kT/n)\ln Z^{(n)}.$$

The integral over $\{K\}$ in Eq. (4) can be carried out formally for independent interactions and the result is

$$Z^{(n)}=\mathrm{Tr}_s e^{-\tilde{\mathcal{H}}/kT}, \qquad (6a)$$

$$\frac{\tilde{\mathcal{H}}}{kT}=-\sum_{k=1}^{\infty}\sum_{\vec{r},\vec{\delta}}C_k\left[\sum_{\alpha=1}^{n}\vec{S}^\alpha(\vec{r})\cdot\vec{S}^\alpha(\vec{r}+\vec{\delta})\right]^k, \qquad (6b)$$

where C_k is the kth cumulant of the distribution $P(K)$: $C_1=[K]_{\mathrm{av}}$, $C_2=[K^2]_{\mathrm{av}}-[K]_{\mathrm{av}}^2$.

For $C_1\neq 0$, the model described above has been used to study the critical properties of dilute quenched random systems.[13] We now consider the spin-glass in which $C_1=0$. For simplicity, we consider first the case of a Gaussian random distribution for which $C_k=0$ for all $k>2$. In this case Eq. (6b) simplifies and $\tilde{\mathcal{H}}/kT$ becomes

$$\sum_{\vec{r},\vec{\delta}}C_2\sum_{\alpha\neq\beta}\sum_{i,j}Q_{ij}^{\alpha\beta}(\vec{r})Q_{ij}^{\alpha\beta}(\vec{r}+\vec{\delta})+\sum_{\vec{r},\vec{\delta}}C_2\sum_{\alpha}[\vec{S}^\alpha(\vec{r})\cdot\vec{S}^\alpha(\vec{r}+\vec{\delta})]^2, \qquad (7)$$

where $\overleftrightarrow{Q}\equiv Q_{ij}^{\alpha\beta}(\vec{r})=S_i^\alpha(\vec{r})S_j^\beta(\vec{r})(1-\delta_{\alpha\beta})$. In the limit $n\to 0$ the order parameter $\langle Q_{ij}^{\alpha\beta}\rangle$ is directly related to the order parameter q of EA:

$$q=\sum_{i=1}^{m}\langle Q_{ii}^{\alpha\beta}\rangle,\quad \alpha\neq\beta. \qquad (8)$$

To construct a field-theoretic formulation we need to express $Z^{(n)}$ in terms of the tensor $\overleftrightarrow{Q}$ which orders at the spin-glass transition. Accordingly, we introduce a probability distribution for $\overleftrightarrow{Q}$ via

$$P(\{\overleftrightarrow{Q}\})=\mathrm{Tr}_s\prod_{\vec{r},\alpha,\beta}\delta(\overleftrightarrow{Q}^{\alpha\beta}(\vec{r})-\vec{S}^\alpha(\vec{r})\vec{S}^\beta(\vec{r}))\exp\{C_2\sum_{\vec{r},\vec{\delta}}\sum_{\alpha}[\vec{S}^\alpha(\vec{r})\cdot\vec{S}^\alpha(\vec{r}+\vec{\delta})]^2\}. \qquad (9)$$

The trace over $\vec{S}$ in Eq. (9) could be performed explicitly. However, since we wish to develop a continuum theory, we observe that the following form for $P(\overleftrightarrow{Q}(\vec{r}))$ will reproduce the constraint that $Q_{ij}^{\alpha\beta}(\vec{r})$ be obtained from spins of unit length according to the definition below Eq. (7):

$$P(\overleftrightarrow{Q}(\vec{r}))\sim\exp[a\,\mathrm{Tr}\overleftrightarrow{Q}^2+w\,\mathrm{Tr}\overleftrightarrow{Q}^3-b\,(\mathrm{Tr}\overleftrightarrow{Q}^2)^2], \qquad (10)$$

where

$$\mathrm{Tr}\overleftrightarrow{Q}^2=\sum_{\alpha,\beta,i,j}Q_{ij}^{\alpha\beta}Q_{ji}^{\beta\alpha},$$

etc. This equation does not include nonlocal effects present in Eq. (9). These will contribute to the $\mathrm{Tr}(\nabla\overleftrightarrow{Q})^2$ term in Eq. (11b). Following the approach used by Wilson and Kogut[14] for the Ising model, we note that if a and b tend to infinity in the appropriate ratio, then the normalization of $\mathrm{Tr}\overleftrightarrow{Q}^2$ is fixed. Furthermore, for fixed $\mathrm{Tr}\overleftrightarrow{Q}^2=n(n-1)$, $\mathrm{Tr}\overleftrightarrow{Q}^3$ is a maximum when $Q_{ij}^{\alpha\beta}$ is of the form $S_i^\alpha S_j^\beta$ with $\vec{S}^\alpha\cdot\vec{S}^\alpha=1$. Thus in the limit when w, a, and b become infinite, in the appropriate way Eq. (10) is valid. A similar scheme has been used by Priest and Lubensky[15] for generalizations of the Potts model. Using Eq. (10) and taking the continuum limit we obtain

$$Z^{(n)} = \int \mathfrak{D}Q\, e^{-\mathcal{H}} \tag{11a}$$

$$\mathcal{H} = \int d^d\vec{x}\{\tfrac{1}{4}r\,\mathrm{Tr}\overleftrightarrow{Q}^2 + \tfrac{1}{4}\mathrm{Tr}(\nabla\overleftrightarrow{Q})^2 - w\,\mathrm{Tr}\overleftrightarrow{Q}^3 + u\,(\mathrm{Tr}\overleftrightarrow{Q}^2)^2 + \textstyle\sum_i v_i F_i(\overleftrightarrow{Q})\}, \tag{11b}$$

where $r \sim T - T^*$, and $T^* \sim [(\delta J)^2]_{av}^{1/2}$ is the mean-field transition temperature. The $F_i(\overleftrightarrow{Q})$ are the fourth-order invariants other than $(\mathrm{Tr}\overleftrightarrow{Q}^2)^2$ which are generated by repeated iterations of the RG. For the $m=1$, Ising case $F_1(\overleftrightarrow{Q}) = \mathrm{Tr}\overleftrightarrow{Q}^4$, $F_2(\overleftrightarrow{Q}) = \sum_{\alpha\beta\delta}(Q_{11}{}^{\alpha\beta})^2(Q_{11}{}^{\alpha\delta})^2$, and $F_3(\overleftrightarrow{Q}) = \sum_{\alpha\beta}(Q_{11}{}^{\alpha\beta})$.[4] For $m \geq 2$ there are many F_i's. The Hamiltonian of Eq. (11b) may also be used when $P(J)$ is non-Gaussian provided $[J]_{av} = 0$. In this case the initial values of the potentials will depend on the details of $P(J)$.

The mean-field minima of Eq. (11b) with $v_i = 0$ can be located by setting $Q_{ij}{}^{\alpha\beta} = m^{-1}q\delta_{ij}I_{\alpha\beta}$, where $I_{\alpha\beta}$ is a symmetric off-diagonal tensor with unit entries. Then we have

$$\frac{m^2\mathcal{H}}{n(n-1)\Omega kT} = \tfrac{1}{4}mrq^2 - w(n-2)q^3 + n(n-1)uq^4, \tag{12}$$

where Ω is the volume. The extrapolation of this mean-field Hamiltonian into the regime $n<1$ is ambiguous, as may also be the case in Ref. 6. There is, however, no ambiguity for $n>1$. We, therefore, calculate physical quantities such as susceptibilities or specific heats, etc., for $1<n<2$ and analytically continue the results to $n=0$. In so doing, no anomalies as a function of n are encountered. For instance, consider the calculation of the order parameter. Remembering that q must be positive, we see[8,15] that Eq. (12) predicts a first-order transition whenever $(n-2)w>0$ and a second-order transition when $(n-2)w<0$. For $w>0$ and $n<2$ one has a second-order transition with $q = mr/[6(n-2)w]$ and we believe this result can be extended to $n \to 0$.

We now discuss the ϵ expansion in $6-\epsilon$ dimensions. The recursion relations are obtained in the standard way and in the notation of Ref. 12 are

$$r' = b^{2-\eta}\{r - 36(n-2)mw^2[A(0) - 2K_6 r\ln b]\}, \tag{13a}$$

$$w' = b^{\epsilon/2 - 3\eta/2}\{w + 36[(n-3)m+1]w^3K_6\ln b\}, \tag{13b}$$

$$\eta = 12(n-2)mw^2K_6. \tag{13c}$$

These equations have a stable fixed point with $(w^*)^2 = -\epsilon\{36K_6[(n-4)m+2]\}^{-1}$ whenever $n<4-(2/m)$. The exponents for $n\to 0$ are listed in Table I.

An interesting possibility exists if $P(J)$ is not symmetric in J. If $[J^3]_{av}$ is nonvanishing, there will be an extra term in the Hamiltonian proportional to $-[J^3]_{av}\mathrm{Tr}\overleftrightarrow{Q}^3$. Thus, if the distribution is sufficiently skewed to the antiferromagnetic side, the sign of w in Eq. (11) can change. In this case, there would be a first-order transition within mean-field theory for $n<2$. Thus the point $w=0$ would correspond to a tricritical point with the skewness of the distribution acting as the nonordering field. The tricritical exponents can be determined from the recursion relations in $4-\epsilon$ dimensions with $w=0$. To first order in ϵ they are of the form

$$u' = b^{\epsilon}[u - 128u^2K_d\ln b - \textstyle\sum_i A_i uv_i - \sum B_{ij}v_iv_j], \tag{14a}$$

$$v_i' = b^{\epsilon}[v_i - 192uv_iK_d\ln b - \textstyle\sum_{ij}C_{ij}v_iv_j], \tag{14b}$$

$$r' = b^2[r - 32ruK_d\ln b + Du], \tag{14c}$$

where A_i, B_{ij}, C_{ij}, and D are constants of order $\ln b$. Thus, at the "Heisenberg" fixed point (HFP) with $u^* = \epsilon/(128K_d)$, $v_i^* = 0$ for all i, one has λ exponents[12] $\lambda_u = -\epsilon$, $\lambda(v_i) = -\frac{1}{2}\epsilon$, for all i, so that the HFP is stable. The other fixed points are assumed to be unstable or unphysical. Also, to lowest order in ϵ near the HFP, one has

$$w' = b^{1+\epsilon/2}w[1 - 96K_4u\ln b], \tag{15}$$

from which one can determine the crossover exponent φ_w defined so that the free energy depends on the variable w as w/r^{φ_w}. The tricritical ex-

TABLE I. Values of exponents correct to first order in ϵ. Other exponents are obtained by $\alpha = 2 - d\nu$, $\beta = \frac{1}{2}(d+\eta-2)\nu$, $\gamma = (2-\eta)\nu$.

Exponent	Critical point at $d=6-\epsilon$	Tricritical point at $d=4-\epsilon$
ν	$\frac{1}{2} + 5m\epsilon/12(2m-1)$	$\frac{1}{2} - \epsilon/16$
η	$-m\epsilon/3(2m-1)$	0
φ_W		$\frac{1}{2} - \frac{3}{16}\epsilon$

ponents deduced from Eqs. (14) and (15) are listed in Table I.

There are other cooperative phenomena which must be re-examined in light of the possible existence of the spin-glass state. In particular we are now considering the competition between ferromagnetic or antiferromagnetic ordering and spin-glass ordering that will occur at some critical value of $[J]_{av}$. We are also considering the possibility of other types of spin-glass ordering that may occur in other types of random systems, e.g., systems with random uniaxial anisotropy.[16]

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Temperature Dependence of Electric Field Gradients in Noncubic Metals*

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Temperature dependence of the nuclear quadrupole frequency, ν_Q, of noncubic metals has been studied theoretically. It is shown that the electronic contribution to the field gradient is largely responsible for the observed $T^{3/2}$ behavior. The conclusions are general and apply to all noncubic metals.

The study of electric field gradients, eq, in metals is of great importance since it not only provides a detailed knowledge of the electronic wave functions in the occupied Fermi volume, but can also yield valuable information regarding the nuclear quadrupole moment, Q. Experiments using a variety of probes, such as nuclear magnetic resonance, time-differential perturbed angular correlation, and Mössbauer effect, have been performed on both pure noncubic metals and alloys to study the distribution of eq. In several systems the sign of the nuclear quadrupole coupling, $\nu_Q = e^2qQ/h$, has also been determined.

Recently the temperature dependence of ν_Q of several metals, such as Cd,[1,2] Zn,[3] In,[4] Sb,[5] and Ga,[6] has been studied experimentally. An analysis of these results reveals the interesting feature that ν_Q generally decreases[7] as $T^{3/2}$ for all these metals, namely,

$$\nu_Q \simeq \nu_Q^0(1 - \alpha T^{3/2}), \tag{1}$$

where ν_Q^0 is the value of the nuclear quadrupole frequency at $T = 0°K$ and α is a constant. Since the electronic structures of all these metals are very different from each other, this "universal" form of the temperature dependence suggests that